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A Class of Nonsymmetric Orthogonal Polynomials on the Unit Circle

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We investigate a particular quadratic decomposition for sequences of orthogonal polynomials, related to quasi-definite functionals on the unit circle. A constructive method is analyzed in order to generate nonsymmetric orthogonal polynomials.

Key Words: orthogonal polynomials; reflection parameters; Caratheódory functions.

1. INTRODUCTION

Let $M=[c_{i-j}]_{i,j=0}^{\infty}$ be an Hermitian Toeplitz matrix, i.e., $c_{-k}=\bar{c}_k$. We will denote by M_n the principal submatrix of size $n+1$. We will assume $\Delta_n=\det M_n\neq 0$ for every $n=0, 1, 2, \ldots$.

It is well known [4] that the sequence of monic polynomials $(\Phi_n)_{n=0}^{\infty}$ given by

$$\Phi_n(z)=\frac{1}{\Delta_{n-1}}\begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ \overline{c_1} & c_0 & \cdots & c_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{c}_{n-1} & \bar{c}_{n-2} & \cdots & c_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \qquad n=1, 2, \ldots,$$

$$\Phi_0(z)=1$$

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is a sequence of monic orthogonal polynomials with respect to the inner product in $\mathbb{P}$, the linear space of polynomials with complex coefficients,

$$(p, q) = \left\langle \mathscr{L}, p(z)\, \bar{q}\left(\frac{1}{z}\right)\right\rangle. \tag{1}$$

Here $\mathscr{L}$ is the linear functional defined on the linear space of Laurent polynomials L in the following way

$$\langle \mathscr{L}, z^n \rangle = c_n, \qquad n = 0, 1, 2, ...,$$
$$\langle \mathscr{L}, z^{-n} \rangle = \overline{c_n}, \qquad n = 0, 1, 2, ...$$

Notice that $L = span\{z^n\}_{n=-\infty}^{\infty}$ and $\mathbb{P} \subset L$.

If $\Delta_n > 0$, $n = 0, 1, ...,$ the linear functional $\mathscr{L}$ is said to be a positive definite linear functional. In such a case, there exists a finite positive Borel measure μ supported on $[-\pi, \pi)$, such that

$$\langle \mathscr{L}, p \rangle = \int_{-\pi}^{\pi} p(e^{i\theta})\, d\mu(\theta).$$

Taking into account (1) it is straightforward to deduce that the shift operator is isometric with respect to (1), i.e.,

$$(zp, zq) = (p, q), \qquad p, q \in \mathbb{P}.$$

As a consequence of this fact, we can deduce two equivalent ways to generate the sequence of monic orthogonal polynomials (SMOP) (Φ_n). They were obtained by Szegő [10] in the positive definite case and by Geronimus [4] in the general situation stated above:

Forward recurrence relation,

$$\begin{aligned} &\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\, \Phi^*_{n-1}(z), \qquad n = 1, 2, ... \\ &\Phi_0(z) = 1, \end{aligned} \tag{2}$$

Backward recurrence relation,

$$\begin{aligned} &\Phi_n(z) = (1 - |\Phi_n(0)|^2)\, z\Phi_{n-1}(z) + \Phi_n(0)\, \Phi^*_n(z), \qquad n = 1, 2, ... \\ &\Phi_0(z) = 1, \end{aligned} \tag{3}$$

where

$$\Phi^*_n(z) = z^n\, \overline{\Phi_n(1/\bar{z})}$$

is the so-called reversed polynomial of Φ_n. The values $\Phi_n(0)$ are called the reflection (or Schur) parameters for the linear functional $\mathscr{L}$. A straightforward computation yields

$$1-|\Phi_n(0)|^2=\frac{\Delta_n\,\Delta_{n-2}}{\Delta_{n-1}^2}.$$

We will denote $k_n^2=(\Phi_n,\Phi_n)=\Delta_n/\Delta_{n-1}=e_n$.

Thus, in the positive definite case $|\Phi_n(0)|<1$ while in the general case considered in this section, $|\Phi_n(0)|\neq 1$.

Conversely, given a sequence of complex numbers $(a_n)_{n=0}^{\infty}$ with $|a_n|\neq 1$, $n=1, 2, \ldots$, and $a_0=1$, there exists a linear functional $\mathscr{L}$ such that $(a_n)_{n=0}^{\infty}$ is the sequence of reflection parameters for the functional, or, equivalently, $a_n=\Phi_n(0)$ where (Φ_n) is the corresponding sequence of monic orthogonal polynomials with respect to $\mathscr{L}$. This result is an analog of Favard's theorem. (See [3] for the positive definite case and [4, Theorem 4.1] for the general case.)

According to this last result, there exists a linear functional $\tilde{\mathscr{L}}$, or equivalently, a sequence of monic orthogonal polynomials (Ω_n) associated with $\tilde{\mathscr{L}}$, such that $\Omega_n(0)=-a_n$, $n=1, 2, \ldots$.

(Ω_n) is called the SMOP of the second kind associated with $\mathscr{L}$.

These polynomials can be explicitly given by

$$\Omega_n(z)=\frac{1}{c_0}\left\langle \mathscr{L}, \frac{y+z}{y-z}\,[\Phi_n(y)-\Phi_n(z)]\right\rangle,$$

where $\mathscr{L}$ acts on the variable y in the right hand side polynomial in two variables.

Finally, we can associate with the linear functional $\mathscr{L}$ a formal series

$$F(z):=c_0+2\sum_{n=1}^{\infty}\bar{c}_n z^n. \tag{4}$$

In the positive definite case, F is an analytic function in the unit disk and $\operatorname{Re} F(z)\geqslant 0$. In the literature, F is said to be a Carathéodory function or C-function. (See [7].) The connection with Schur functions and the Schur algorithm is analyzed in [1].

The link between the formal series F and the sequences (Φ_n) and (Ω_n) is the following

THEOREM 1 [8]. *The sequence of monic orthogonal polynomials* (Φ_n), *the corresponding polynomials of the second kind* (Ω_n), *and the Carathéodory function* F, *satisfy the relation*

$$\Phi_n(z)\,F(z)+\Omega_n(z)=O(z^n),$$
$$\Phi_n^*(z)\,F(z)-\Omega_n^*(z)=O(z^{n+1})$$

for $z\to 0$.

In comparison with the real case (see [2]), very few explicit examples of SMOP with respect to a linear functional are known in the literature.

A way to generate a new SMOP from a given SMOP (Φ_n) is to consider a sieving process. Unfortunately, there is a strong constraint. For instance —and this is a basic difference with the real case—there exists a unique SMOP (Ψ_n) such that $\Psi_{2n+1}(z)=z\Phi_n(z^2)$. Furthermore, $\Psi_{2n}(z)=\Phi_n(z^2)$. See [5, 6].

In terms of the reflection parameters, this means that the linear transform T in the space of the sequences of reflection parameters is given by

$$T(a_{2n})=a_n,\qquad n=0,1,2,...$$
$$T(a_{2n+1})=0,\qquad n=0,1,2,...$$

with $\Phi_n(0)=a_n$.

For the corresponding formal series,

$$F_T(z)=F(z^2)$$

holds.

The aim of our contribution is the analysis of necessary and sufficient conditions in order that a sequence of polynomials $(\tilde{\Psi}_n)$ defined as a perturbation of (Ψ_n) in the following way

$$\begin{aligned}\tilde{\Psi}_{2n}(z)&=\Phi_n(z^2)+zB_{n-1}(z^2)\\ \tilde{\Psi}_{2n+1}(z)&=z\Phi_n(z^2)+D_n(z^2)\end{aligned}\tag{5}$$

with $n=0,1,...,B_{-1}\equiv 0$, and $\deg B_{n-1}\leqslant n-1$, $\deg D_n\leqslant n$, be an SMOP. Next we deduce the expression of the formal series $\tilde{F}$ in terms of the formal series F corresponding to the linear functional $\mathscr{L}$ and the relation between the corresponding Szegő functions. The particular situation when $\mathscr{L}$ is a positive definite functional will be considered. Finally we illustrate the preceding with some examples: the case of real Schur parameters and Bernstein–Szegő polynomials.

Our results are a continuation of work started in [6], which is the unit circle analog of a problem raised and solved in the real case by T. S. Chihara and L. Chihara [2].

2. CONDITIONS FOR ORTHOGONALITY

THEOREM 2. *Suppose that an SMOP* (Φ_n) *is given. Then the sequence* $(\tilde{\Psi}_n)$, *defined by* (5) *is an SMOP if and only if* $D_n(0) \neq 0$ *for at most one* $n \in \{0, 1, 2, ...\}$, *and the polynomials* (B_n) *and* (D_n) *satisfy*:

(a) *if* $D_n(0) = 0$ *for all* $n = 0, 1, 2, ...$, *then* $B_n(z) = D_n(z) = 0$ *for all* $n = 0, 1, ...$;

(b) *if* $D_N(0) \neq 0$, *then* $B_n(z) = D_n(z) = 0$ *for* $n = 0, 1, ..., N-1$,

$$D_N(z) = D_N(0)\, \Phi_N^*(z), \qquad B_N(z) = D_N(z) + \Phi_{N+1}(0)\, D_N^*(z),$$

$D_n(z) = zB_{n-1}(z)$ *for* $n \geqslant N+1$, *and*

$$B_{n+1}(z) = zB_n(z) + \Phi_{n+2}(0)\, B_n^*(z), \qquad n \geqslant N.$$

Proof. If $(\tilde{\Psi}_n)$ is an SMOP, then the forward recurrence relation

$$\tilde{\Psi}_{2n}(z) = z\tilde{\Psi}_{2n-1}(z) + \tilde{\Psi}_{2n}(0)\, \tilde{\Psi}_{2n-1}^*(z), \qquad n = 1, 2, ...$$

Together with (5) gives

$$\Phi_n(z^2) + zB_{n-1}(z^2) = z^2\Phi_{n-1}(z^2) + \Phi_n(0)\, \Phi_{n-1}^*(z^2)$$
$$+ zD_{n-1}(z^2) + \Phi_n(0)\, zD_{n-1}^*(z^2), \qquad n = 1, 2, ...$$

Since (Φ_n) is an SMOP, we have $\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\, \Phi_{n-1}^*(z)$, so that

$$B_{n-1}(z) = D_{n-1}(z) + \Phi_n(0)\, D_{n-1}^*(z), \qquad n = 1, 2, ..., \tag{6}$$

where $D_n^*(z) = z^n\, \overline{D_n(1/\bar{z})}$, even if $\deg D_n \leqslant n$.

On the other hand, from

$$\tilde{\Psi}_{2n+1}(z) = z\tilde{\Psi}_{2n}(z) + \tilde{\Psi}_{2n+1}(0)\, \tilde{\Psi}_{2n}^*(z), \qquad n = 0, 1, 2, ...$$

the relation (5) gives

$$z\Phi_n(z^2) + D_n(z^2) = z\Phi_n(z^2) + z^2B_{n-1}(z^2)$$
$$+ D_n(0)\, \Phi_n^*(z^2) + D_n(0)\, zB_{n-1}^*(z^2), \qquad n = 0, 1, 2, ...,$$

where $B_n^*(z) = z^n \overline{B_n(1/\bar{z})}$, even if $\deg B_n \leqslant n$. Hence

$$D_n(z^2) = z^2 B_{n-1}(z^2) + D_n(0)\, \Phi_n^*(z^2) + D_n(0)\, z B_{n-1}^*(z^2).$$

Thus, since $D_n(z^2)$ is an even polynomial,

$$\begin{aligned} D_0(0) &= \tilde{\Psi}_1(0) \\ D_n(0)\, B_{n-1}^*(z) &= 0, \qquad n = 1, 2, ..., \end{aligned} \tag{7}$$

so that

$$D_n(z) = z B_{n-1}(z) + D_n(0)\, \Phi_n^*(z) \tag{8}$$

for all $n = 0, 1, 2, ...$

We will consider two possible situations:

(i) $D_n(0) = 0$ for every $n = 0, 1, 2, ...$ Then, from (8)

$$D_n(z) = z B_{n-1}(z)$$

for $n = 0, 1, 2, ...$ Furthermore, in (6) we get

$$B_{n-1}(z) = z B_{n-2}(z) + \Phi_n(0)\, B_{n-2}^*(z), \qquad n = 1, 2, 3, ...$$

Substituting $n = 1$ in (6) gives

$$B_0(z) = 0.$$

With this initial condition it follows that $B_n(z) = 0$ for every $n = 0, 1, ...$ and $D_n(z) = 0$ for every $n = 0, 1, ...$

In conclusion, we have in this case

$$\begin{aligned} \tilde{\Psi}_{2n}(z) &= \Phi_n(z^2), \\ \tilde{\Psi}_{2n+1}(z) &= z \Phi_n(z^2). \end{aligned}$$

(ii) $D_n(0) \neq 0$ for at least one $n \in \mathbb{N}$. Let N be a fixed nonnegative integer and assume $D_N(0) \neq 0$. If $N \geqslant 1$, then from (7), it follows that $B_{N-1}(z) = 0$. Using (6) $D_{N-1}(z) = 0$. Again from (8), $B_{N-2}(z) = 0$. Repeating the process gives $B_k(z) = D_k(z) = 0$ for $k = 0, 1, ..., N-1$. If there exists $M > N$ such that $D_M(0) \neq 0$, we get a contradiction because in such a case, according to the above reasoning $D_N(0)$ must vanish.

Thus two cases appear:

(a) $D_0(0) \neq 0$ and $D_n(0) = 0$, $n = 1, 2, \ldots$. Then

$$D_0(z) = D_0(0), \qquad \alpha = B_0(z) = D_0(0) + \Phi_1(0)\,\overline{D_0(0)} \neq 0,$$
$$D_n(z) = zB_{n-1}(z), \qquad n = 1, 2, \ldots,$$
$$B_n(z) = zB_{n-1}(z) + \Phi_{n+1}(0)\,B^*_{n-1}(z), \quad n = 1, 2, \ldots$$

This last relation means that the sequence of monic polynomials (V_n) with $V_n(z) = B_n(z)/B_0(z)$ satisfies a forward recurrence relation as (2) with reflection parameters $V_n(0) = e^{i\varphi}\Phi_{n+1}(0)$, $n = 1, 2, \ldots$, and $e^{i\varphi} = \overline{B_0}(z)/B_0(z)$. This corresponds to a shift in the sequence of reflection parameters and (V_n) is again a sequence of monic orthogonal polynomials. In fact

$$V_n(z) = \frac{p(z)\,\Phi_{n+1}(z) + q(z)\,\Omega_{n+1}(z)}{2z(1 - |\Phi_1(0)|^2)}, \tag{9}$$

where

$$p(z) = (e^{i\varphi} - \overline{\Phi_1(0)})\,z + (1 - e^{i\varphi}\Phi_1(0))$$
$$q(z) = (\overline{\Phi_1(0)} - e^{i\varphi})\,z + (1 - e^{i\varphi}\Phi_1(0))$$

(see [7]).

Thus

$$\tilde{\Psi}_{2n}(z) = \Phi_n(z^2) + \alpha z V_{n-1}(z^2), \qquad n = 0, 1, 2, \ldots$$
$$\tilde{\Psi}_{2n+1}(z) = z\tilde{\Psi}_{2n}(z), \qquad n = 1, 2, \ldots$$

while $\tilde{\Psi}_1(z) = z + D_0(0)$.

(b) $D_N(0) \neq 0$, $N \geqslant 1$, and $D_n(0) = 0$ for $n \neq N$. Then $B_m(z) = D_m(z) = 0$, $m = 0, 1, \ldots, N-1$. On the other hand, from (8)

$$D_N(z) = D_N(0)\,\Phi^*_N(z).$$

From (6)

$$B_N(z) = D_N(z) + \Phi_{N+1}(0)\,D^*_N(z).$$

But from (8)

$$D_n(z) = zB_{n-1}(z), \qquad n = N+1, \ldots$$

and by substitution in (6)

$$B_{n+1}(z) = zB_n(z) + \Phi_{n+2}(0)\,B^*_n(z), \qquad n = N, N+1, \ldots. \tag{10}$$

Taking into account that the leading coefficient of B_N is

$$\alpha_N = D_N(0)\,\overline{\Phi_N(0)} + \Phi_{N+1}(0)\,\overline{D_N(0)}$$

we get

(b.1) If $\Phi_N(0) \neq 0$, i.e., $\alpha_N \neq 0$, then $\deg B_n = n$ for $n = N, N+1, \ldots,$ and the sequence $(B_n, n \geqslant N)$ can be obtained explicitly from (10).

(b.2) If $\Phi_N(0) = 0$, then $\deg B_N = k \leqslant N-1$. Thus, $\deg B_n = k+n-N$ for every $n = N, N+1, \ldots,$ and the sequence $(B_n, n \geqslant N)$ can be obtained explicitly from (10).

In both cases

$$\begin{cases} \tilde{\Psi}_{2n}(z) = \Phi_n(z^2), & n = 0, 1, \ldots, N \\ \tilde{\Psi}_{2n-1}(z) = z\Phi_{n-1}(z^2), & n = 1, 2, \ldots, N \end{cases}$$

$$\tilde{\Psi}_{2N+1}(z) = z\Phi_N(z^2) + D_N(0)\,\Phi_N^*(z^2)$$

$$\begin{cases} \tilde{\Psi}_{2n+2}(z) = \Phi_{n+1}(z^2) + zB_n(z^2), & n = N, N+1, \ldots \\ \tilde{\Psi}_{2n+3}(z) = z\tilde{\Psi}_{2n+2}(z), & n = N, N+1, \ldots \end{cases}$$

with

$$B_n(z) = zB_{n-1}(z) + \Phi_{n+1}(0)\,B_{n-1}^*(z)$$

for $n \geqslant N+1$ and

$$B_N(z) = D_N(0)\,\Phi_N^*(z) + \Phi_{N+1}(0)\,\overline{D_N(0)}\,\Phi_N(z). \quad \blacksquare \tag{11}$$

As a conclusion, if $|\gamma| \neq 1$, where $\gamma = D_N(0)$, there exists a unique SMOP $(\tilde{\Psi}_n)$ such that the reflection parameters are

$$\begin{cases} \tilde{\Psi}_{2n}(0) = \Phi_n(0), & n = 0, 1, \ldots, \\ \tilde{\Psi}_{2n+1}(0) = 0, & n \neq N, \\ \tilde{\Psi}_{2N+1}(0) = \gamma. \end{cases} \tag{12}$$

Remark. Assume that α is a zero of B_N, i.e., $B_N(\alpha) = 0$, $(N \geqslant 1)$. Then we have

$$|D_N(\alpha)| = |\Phi_{N+1}(0)|\,|D_N^*(\alpha)|$$

and from (11)

$$|D_N(0)|\,|\Phi_N^*(\alpha)| = |\Phi_{N+1}(0)|\,|\overline{D_N(0)}|\,|\Phi_N(\alpha)|.$$

Thus, in the positive definite case

$$\frac{|\Phi_N^*(\alpha)|}{|\Phi_N(\alpha)|} < 1,$$

so that $|\alpha| > 1$. So the monic polynomials corresponding to (B_n) cannot be an SMOP.

The next step will be an alternative way to deduce an explicit expression for the sequence $(\tilde{\Psi}_n)$ in terms of the sequences (Φ_n) and (Ω_n), where (Ω_n) is the SMOP of the second kind for (Φ_n).

Taking into account (11), $(\tilde{\Psi}_n)$ is a finite perturbation of (Ψ_n) at level $2N+1$, i.e., the reflection parameters of these two SMOP are the same for $m \geqslant 2N+2$ (and in our case also coincide for $m \leqslant 2N$).

Thus,

$$\begin{cases} \tilde{\Psi}_{2n}(z) = \Phi_n(z^2), & n = 0, 1, ..., N \\ \tilde{\Psi}_{2n+1}(z) = z\Phi_n(z^2), & n = 0, 1, ..., N-1 \\ \tilde{\Psi}_{2N+1}(z) = z\Phi_N(z^2) + \gamma\Phi_N^*(z^2). \end{cases}$$

For the remaining terms we have, taking into account Theorem 3.1 in [7],

$$\frac{[\tilde{\Lambda}_{2N+1}(z) + \tilde{\Lambda}_{2N+1}^*(z)]\, \tilde{\Psi}_{2N+1+k}(z) + [\tilde{\Psi}_{2N+1}^*(z) - \tilde{\Psi}_{2N+1}(z)]\, \tilde{\Lambda}_{2N+1+k}(z)}{1-|\gamma|^2}$$

$$= [\Lambda_{2N+1}(z) + \Lambda_{2N+1}^*(z)]\, \Psi_{2N+1+k}(z) + [\Psi_{2N+1}^*(z) - \Psi_{2N+1}(z)]\, \Lambda_{2N+1+k}(z)$$

and

$$\frac{[\tilde{\Psi}_{2N+1}(z) + \tilde{\Psi}_{2N+1}^*(z)]\, \tilde{\Lambda}_{2N+1+k}(z) + [\tilde{\Lambda}_{2N+1}^*(z) - \tilde{\Lambda}_{2N+1}(z)]\, \tilde{\Psi}_{2N+1+k}(z)}{1-|\gamma|^2}$$

$$= [\Psi_{2N+1}(z) + \Psi_{2N+1}^*(z)]\, \Lambda_{2N+1+k}(z) + [\Lambda_{2N+1}^*(z) - \Lambda_{2N+1}(z)]\, \Psi_{2N+1+k}(z),$$

where $(\tilde{\Lambda}_n)$ and (Λ_n) are, respectively, the sequences of monic polynomials of the second kind associated with $(\tilde{\Psi}_n)$ and (Ψ_n).

In matrix form

$$\frac{1}{1-|\gamma|^2}\begin{pmatrix} \tilde{\Lambda}_{2N+1}(z) & -\tilde{\Psi}_{2N+1}(z) \\ \tilde{\Lambda}_{2N+1}^*(z) & \tilde{\Psi}_{2N+1}^*(z) \end{pmatrix}\begin{pmatrix} \tilde{\Psi}_{2N+1+k}(z) \\ \tilde{\Lambda}_{2N+1+k}(z) \end{pmatrix}$$

$$= \begin{pmatrix} \Lambda_{2N+1}(z) & -\Psi_{2N+1}(z) \\ \Lambda_{2N+1}^* & \Psi_{2N+1}^*(z) \end{pmatrix}\begin{pmatrix} \Psi_{2N+1+k}(z) \\ \Lambda_{2N+1+k}(z) \end{pmatrix}.$$

Keeping in mind that

$$\Psi_{2N}(z) = \Phi_N(z^2), \qquad \Lambda_{2N}(z) = \Omega_N(z^2),$$
$$\Psi_{2N+1}(z) = z\Phi_N(z^2), \qquad \Lambda_{2N+1}(z) = z\Omega_N(z^2),$$

and

$$\tilde{\Psi}_{2N+1}(z) = z\tilde{\Psi}_{2N}(z) + \gamma\tilde{\Psi}^*_{2N}(z),$$
$$\tilde{\Lambda}_{2N+1}(z) = z\tilde{\Lambda}_{2N}(z) - \gamma\tilde{\Lambda}^*_{2N}(z),$$

we then find

$$\begin{aligned}
&\begin{pmatrix} \tilde{\Psi}_{2N+1+k}(z) \\ \tilde{\Lambda}_{2N+1+k}(z) \end{pmatrix} \\
&= \begin{pmatrix} z\Omega_N(z^2) & -z\Phi_N(z^2) \\ \Omega^*_N(z^2) & \Phi^*_N(z^2) \end{pmatrix}^{-1} \begin{pmatrix} 1 & \gamma \\ \bar{\gamma} & 1 \end{pmatrix} \begin{pmatrix} z\Omega_N(z^2) & -z\Phi_N(z^2) \\ \Omega^*_N(z^2) & \Phi^*_N(z^2) \end{pmatrix} \begin{pmatrix} \Psi_{2N+1+k}(z) \\ \Lambda_{2N+1+k}(z) \end{pmatrix} \\
&= \frac{1}{z(\Omega_N(z^2)\,\Phi^*_N(z^2) + \Omega^*_N(z^2)\,\Phi_N(z^2))} \begin{pmatrix} \Phi^*_N(z^2) & z\Phi_N(z^2) \\ -\Omega^*_N(z^2) & z\Omega_N(z^2) \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & \gamma \\ \bar{\gamma} & 1 \end{pmatrix} \begin{pmatrix} z\Omega_N(z^2) & -z\Phi_N(z^2) \\ \Omega^*_N(z^2) & \Phi^*_N(z^2) \end{pmatrix} \begin{pmatrix} \Psi_{2N+1+k}(z) \\ \Lambda_{2N+1+k}(z) \end{pmatrix}.
\end{aligned}$$

Taking into account that (see [4])

$$\Omega_N(z)\,\Phi^*_N(z) + \Omega^*_N(z)\,\Phi_N(z) = 2z^N e_N = 2z^N \prod_{i=1}^{N} (1 - |\Phi_i(0)|^2),$$

we get

$$\begin{aligned}
&\begin{pmatrix} \tilde{\Psi}_{2N+1+k}(z) \\ \tilde{\Lambda}_{2N+1+k}(z) \end{pmatrix} \\
&= \frac{1}{2z^{2N+1}e_N} \begin{pmatrix} \Phi^*_N(z^2) + z\bar{\gamma}\Phi_N(z^2) & \gamma\Phi^*_N(z^2) + z\Phi_N(z^2) \\ -\Omega^*_N(z^2) + z\bar{\gamma}\Omega_N(z^2) & -\gamma\Omega^*_N(z^2) + z\Omega_N(z^2) \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} z\Omega_N(z^2) & -z\Phi_N(z^2) \\ \Omega^*_N(z^2) & \Phi^*_N(z^2) \end{pmatrix} \begin{pmatrix} \Psi_{2N+1+k}(z) \\ \Lambda_{2N+1+k}(z) \end{pmatrix} \\
&= \frac{1}{2z^{2N+1}e_N} \begin{pmatrix} R(z) & S(z) \\ U(z) & V(z) \end{pmatrix} \begin{pmatrix} \Psi_{2N+1+k}(z) \\ \Lambda_{2N+1+k} \end{pmatrix}.
\end{aligned} \tag{13}$$

Here

$$R(z)=2z^{2N+1}e_N+z^2\bar{\gamma}\Phi_N(z^2)\,\Omega_N(z^2)+\gamma\Phi_N^*(z^2)\,\Omega_N^*(z^2),$$

$$S(z)=\gamma(\Phi_N^*(z^2))^2-z^2\bar{\gamma}(\Phi_N(z^2))^2,$$

$$U(z)=-\gamma(\Omega_N^*(z^2))^2+z^2\bar{\gamma}(\Omega_N(z^2))^2,$$

$$V(z)=2z^{2N+1}e_N-z^2\bar{\gamma}\Phi_N(z^2)\,\Omega_N(z^2)-\gamma\Phi_N^*(z^2)\,\Omega_N^*(z^2).$$

Thus, for $m=N+1, ...,$

$$\begin{aligned}\tilde{\Psi}_{2m}(z)=\Psi_{2m}(z)+\frac{1}{2e_N}\,z^{2N+1}[\,(\bar{\gamma}z^2\Phi_N(z^2)\,\Omega_N(z^2)\\ +\gamma\Phi_N^*(z^2)\,\Omega_N^*(z^2))\,\Phi_m(z^2)\\ +(\gamma(\Phi_N^*(z^2))^2-\bar{\gamma}z^2(\Phi_N(z^2))^2)\,\Omega_m(z^2)].\end{aligned}$$

In other words

$$\begin{aligned}B_{m-1}(z)=\frac{1}{2e_Nz^{N+1}}\,[\bar{\gamma}z^2\Phi_N(z)\,\Omega_N(z)\\ +\gamma\Phi_N^*(z)\,\Omega_N^*(z))\,\Phi_m(z)+(\gamma(\Phi_N^*(z))^2-\bar{\gamma}z(\Phi_N(z))^2)\,\Omega_m(z)].\end{aligned}$$

If we denote by $\tilde{F}$ the C-function associated with the SMOP $\{\tilde{\Psi}_n\}$, and if we take into account Theorem 1, we get from (13)

Proposition 3. *For the Carathéodory function associated with SMOP* $(\tilde{\Psi}_n)$ *we have the relation*

$$\tilde{F}(z)=\frac{V(z)\,F(z^2)-U(z)}{-S(z)\,F(z^2)+R(z)}.$$

Proof. For $n\geqslant 2N+1$ we have

$$\frac{\Lambda_n^*(z)}{\Psi_n^*(z)}=\frac{-U(z)\,\Psi_n^*(z)+V(z)\,\Lambda_n^*(z)}{R(z)\,\Psi_n^*(z)-S(z)\,\Lambda_n^*(z)}$$

according to (13). Then

$$\tilde{F}(z)=\lim_n\frac{\tilde{\Lambda}_n^*(z)}{\tilde{\Psi}_n^*(z)}=\frac{-U(z)\,F(z^2)+V(z)}{R(z)-S(z)\,F(z^2)}. \quad ∎$$

Recall that, in the positive definite case, the measure $d\mu$ belongs to the Szegő class when $(\Phi_n(0)) \in l_2$. In this case we can define the Szegő function as

$$D(z; d\mu) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} \log \mu'(\theta)\, d\theta\right\}, \qquad |z|<1$$

(see [10]) and furthermore

$$D(z; d\mu) = \lim_n \frac{\kappa_n}{\Phi_n^*(z)} \tag{14}$$

locally uniformly in $|z|<1$.

Proposition 4. *Let $d\mu$ be in the Szegő class. For $\gamma \in \mathbb{C}$, $|\gamma|<1$, there exists a measure $d\tilde{\nu}$ associated with the polynomials $(\tilde{\Psi}_n)$, which belongs to the Szegő class. Moreover, the corresponding Szegő function is*

$$D(z; d\tilde{\nu}) = (1-|\gamma|^2)^{-1/2} \frac{D(z^2, d\mu)}{R(z) - S(z)\, F(z)}, \qquad |z|<1.$$

Proof. According to (14) we have

$$\begin{aligned} D(z; d\tilde{\nu}) &= \lim_n \frac{\tilde{\kappa}_n}{\tilde{\Phi}_n^*(z)} \\ &= \lim_n \frac{(1-|\gamma|^2)^{-1/2}\, \kappa_n}{R(z)\, \Phi_n^*(z^2) - S(z)\, \Omega_n^*(z^2)} \\ &= (1-|\gamma|^2)^{-1/2} \lim_n \frac{\kappa_n/\Phi_n^*(z^2)}{R(z) - S(z)\,(\Omega_n^*(z^2)/\Phi_n^*(z^2))} \\ &= (1-|\gamma|^2)^{-1/2} \frac{D(z^2; d\mu)}{R(z) - S(z)\, F(z^2)} \end{aligned}$$

in $|z|<1$. ∎

3. SOME EXAMPLES

1. *Real Parameters*

We illustrate the preceding results with some examples. First, we consider real Schur parameters $\Phi_n(0) \in \mathbb{R}$ $(n \geqslant 0)$, $|\Phi_n(0)|<1$, as well as $\gamma \in \mathbb{R}$ and $|\gamma|<1$.

The relation between orthogonal polynomials on the unit circle $(\Phi_n)_{n\in\mathbb{N}}$ and orthogonal polynomials $(P_n)_{n\in\mathbb{N}}$ on the real axis is well known. (See [4, 10, 11].)

In fact, if

$$xP_n(x)=P_{n+1}(x)+\beta_n P_n(x)+\gamma_n P_{n-1}(x)$$

then

$$\begin{aligned}2\beta_n&=\Phi_{2n-1}(0)(1-\Phi_{2n}(0))-\Phi_{2n+1}(0)(1+\Phi_{2n}(0)),\\ 4\gamma_{n+1}&=(1-\Phi_{2n+2}(0))(1-\Phi^2_{2n+1}(0))(1+\Phi_{2n}(0)).\end{aligned}$$

If denote by $(\tilde{P}_n)_{n\in\mathbb{N}}$ the SMOP on $\mathbb{R}$ associated to $(\tilde{\Psi}_n)_{n\in\mathbb{N}}$ and by $\tilde{\beta}_n$, $\tilde{\gamma}_n$ respectively the coefficients of the corresponding three-term recurrence relation, we can deduce in a straightforward way that

$$\begin{aligned}\tilde{\beta}_n&=0, \qquad n\neq N, N+1\\ 2\tilde{\beta}_N&=-\gamma(1+\Phi_N(0)),\\ 2\tilde{\beta}_{N+1}&=\gamma(1+\Phi_N(0)),\\ 4\tilde{\gamma}_{n+1}&=(1+\Phi_{n+1}(0))(1+\Phi_n(0)), \qquad n\neq N,\\ 4\tilde{\gamma}_{N+1}&=(1-\Phi_{N+1}(0))(1-\gamma^2)(1+\Phi_N(0)).\end{aligned}$$

We now take into consideration the polynomials $(\tilde{P}_n)$. We can express them in terms of (P_n), via the family $(\bar{P}_n)$, associated with the SMOP (Ψ_n).

For the case $\Phi_n(0)\in\mathbb{R}$ $(n\geqslant 0)$, the relation between Φ_n and P_n can be written (see [10])

$$P_n(x)=\frac{\Phi_{2n}(z)+\Phi^*_{2n}(z)}{(1+\Phi_{2n}(0))\,2^n z^n}=\frac{z\Phi_{2n-1}(z)+\Phi^*_{2n-1}(z)}{2^n z^n},$$

where $x=(z+z^{-1})/2$. In a similar way, for the second kind of polynomials we can define the SMOP $(Q_n)_{n=0}^{\infty}$ (see [7])

$$Q_n(x)=\frac{z\Omega_{2n-1}(z)+\Omega^*_{2n-1}(z)}{2^n z^n},$$

and

$$iyP^{(1)}_{n-1}(x)=\frac{z\Omega_{2n-1}(z)-\Omega^*_{2n-1}(z)}{2^n z^n},$$

$$iyQ^{(1)}_{n-1}(x)=\frac{z\Phi_{2n-1}(z)-\Phi^*_{2n-1}(z)}{2^n z^n},$$

where $(P_n^{(1)})$ and $(Q_n^{(1)})$ are the first kind associated SMOP and $y=(z-z^{-1})/2i$.

Thus

$$\begin{aligned}\Phi_{2n-1}(z)&=2^{n-1}z^{n-1}(P_n(x)+iyQ_{n-1}^{(1)}(x)),\\ \Phi_{2n-1}^*(z)&=2^{n-1}z^{n}(P_n(x)-iyQ_{n-1}^{(1)}(x)),\\ \Omega_{2n-1}(z)&=2^{n-1}z^{n-1}(Q_n(x)+iyP_{n-1}^{(1)}(x)),\\ \Omega_{2n-1}^*(z)&=2^{n-1}z^{n}(Q_n(x)-iyP_{n-1}^{(1)}(x)).\end{aligned} \tag{15}$$

Keeping in mind the definition of $(\Psi_n)_{n=0}^{\infty}$, we get

$$\bar{P}_m(x)=\frac{\Phi_m(z^2)+\Phi_m^*(z^2)}{(1+\Phi_m(0))\,2^m z^m}.$$

For $m=2n$ the above expression becomes

$$\bar{P}_{2n}(x)=\frac{\Phi_{2n}(z^2)+\Phi_{2n}^*(z^2)}{(1+\Phi_{2n}(0))\,2^{2n} z^{2n}}.$$

If we denote $w=z^2$, $u=(w+w^{-1})/2$, $v=(w-w^{-1})/2i$, we have $u=2x^2-1$, $v=2xy$. Then from (15)

$$\bar{P}_{2n}(x)=2^{-n}P_n(2x^2-1).$$

Similarly, for $m=2n+1$,

$$\begin{aligned}\bar{P}_{2n+1}(x)&=\frac{\Phi_{2n+1}(w)+\Phi_{2n+1}^*(w)}{(1+\Phi_{2n+1}(0))\,2^{2n+1}z^{2n+1}}\\ &=\frac{P_{n+1}(u)+ivQ_n^{(1)}(u)+w[P_{n+1}(u)-ivQ_n^{(1)}(u)]}{(1+\Phi_{2n+1}(0))\,2^{n+1}z}\\ &=\frac{xP_{n+1}(2x^2-1)+2x(1-x^2)\,Q_n^{(1)}(2x^2-1)}{2^n(1+\Phi_{2n+1}(0))}.\end{aligned}$$

In both cases, for $n\geqslant N+1$, $\tilde{P}_n$ can be given in terms of $\bar{P}_n$ in the following way.

From (13)

$$\begin{aligned}\tilde{P}_n(x)&=\frac{\tilde{\Psi}_{2n}(z)+\tilde{\Psi}_{2n}^*(z)}{(1+\Psi_{2n}(0))\,2^n z^n}\\ &=\frac{1}{(1+\Psi_{2n}(0))\,2^n z^n}\left[\frac{R(z)\,\Psi_{2n}(z)+S(z)\,\Psi_{2n}^*(z)}{2z^{2N+1}e_N}\right.\\ &\qquad\left.+\frac{R^*(z)\,\Psi_{2n}^*(z)+S^*(z)\,\Psi_{2n}(z)}{2z^{2N+1}e_N}\right].\end{aligned}$$

By considering that $R^*(z) = R(z)$ and $S^*(z) = -S(z)$, we have

$$\tilde{P}_n(x) = \frac{R(z)[\Psi_{2n}(z) + \Psi^*_{2n}(z)] + S(z)[\Lambda_{2n}(z) - \Lambda^*_{2n}(z)]}{(1 + \Psi_{2n}(0))\, 2^n z^n 2z^{2N+1} e_N}$$

$$= \frac{R(z)}{2z^{2N+1} e_N} \bar{P}_n(x) + \frac{S(z)}{2z^{2N+1} e_N} iy \bar{P}^{(1)}_{n-1}(x).$$

Since $\gamma \in \mathbb{R}$, from (13) and (15) we get

$$\begin{aligned} R(z) &= 2z^{2N+1} e_N + \gamma[\Psi_{2N+1}(z) \Lambda_{2N+1}(z) + \Psi^*_{2N+1}(z)\, \Lambda^*_{2N+1}(z)] \\ &= 2z^{2N+1} e_N + 2^{2N} z^{2N+1} \gamma[(\bar{P}_{N+1}(x)\, \bar{Q}_{N+1}(x) \\ &\quad - y^2 \bar{P}^{(1)}_N(x)\, \bar{Q}^{(1)}_N(x))(z + z^{-1}) \\ &\quad - iy(\bar{P}_{N+1}(x)\, \bar{P}^{(1)}_N(x) + \bar{Q}_{N+1}(x)\, \bar{Q}^{(1)}_N(x))(z - z^{-1})]. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{R(z)}{2z^{2N+1} e_N} &= 1 + \frac{2^{2N}\gamma}{e_N} [x(\bar{P}_{N+1}(x)\, \bar{Q}_{N+1}(x) - (1 - x^2)\, \bar{P}^{(1)}_N(x)\, \bar{Q}^{(1)}_N(x)) \\ &\quad + (1 - x^2)(\bar{P}_{N+1}(x)\, \bar{P}^{(1)}_N(x) + \bar{Q}_{N+1}(x)\, \bar{Q}^{(1)}_N(x))] \\ &= \rho(x), \end{aligned}$$

where ρ is a polynomial of degree less than or equal to $2N + 3$.

In an analogous way

$$\begin{aligned} S(z) &= \gamma[\Psi^{*2}_{2N+1}(z) - \Psi^2_{2N+1}(z)] \\ &= 2^{2N} z^{2N+1} \gamma[(\bar{P}^2_{N+1}(x) - y^2 \bar{Q}^{(1)^2}_N(x))(z - z^{-1}) \\ &\quad - 2iy \bar{P}_{N+1}(x)\, \bar{Q}^{(1)}_N(x)(z + z^{-1})]. \end{aligned}$$

Then

$$\frac{S(z)}{2z^{2N+1} e_N} = 2^{2N}\gamma[\bar{P}^2_{N+1}(x) - y^2 \bar{Q}^{(1)^2}_N(x) - 2x \bar{P}_{N+1}(x)\, \bar{Q}^{(1)}_N(x)]\, iy = -iy\sigma(x)$$

with $\deg \sigma \leqslant 2N + 2$. In both cases, $R(z)/2^{2N+1} z^{2N+1}$ and $S(z)/2^{2N+1} z^{2N+1}$ can be expressed in terms of Chebyshev polynomials of the first and second kind

$$\rho(x) = \sum_{j=0}^{N} \alpha_j T_{2N+1-2j}(x), \qquad -iy\sigma(x) = (z - z^{-1}) \sum_{j=0}^{N} \beta_j U_{2N-2j}(x),$$

where α_j and β_j are real coefficients, depending on the Schur parameters $(\Phi_j(0))_{j=0}^{N}$.

Now, we have

$$\tilde{P}_n(x) = \rho(x)\,\bar{P}_n(x) + (1-x^2)\,\sigma(x)\,\bar{P}_n^{(1)}(x).$$

So, we have deduced the relation between $(\tilde{P}_n)$ and (P_n).

2. *Bernstein–Szegő Polynomials*

Now we consider Bernstein–Szegő type polynomials, i.e., the SMOP defined by

$$\Phi_n(0) = 0, \qquad \forall n \geqslant k+1, \quad k \geqslant 0,$$

and $|\Phi_n(0)| < 1$, $n \leqslant k$.

We will maintain the notation of the previous sections. Then, the corresponding C-function is

$$F(z) = \frac{\Omega_k^*(z)}{\Phi_k^*(z)}.$$

Now, we make the perturbation at level $2N+1$ with $N \geqslant k$. Thus

$$\tilde{\Psi}_{2N+1}(z) = z\Psi_{2N}(z) + \gamma \Psi_{2N}^*(z),$$

$$\tilde{\Lambda}_{2N+1}(z) = z\Lambda_{2N}(z) - \gamma \Lambda_{2N}^*(z),$$

$(|\gamma| < 1)$, and for $m \geqslant 2N+1$,

$$\tilde{\Psi}_{m+1}(z) = z^{m-2N}\tilde{\Psi}_{2N+1}(z),$$

$$\tilde{\Lambda}_{m+1}(z) = z^{m-2N}\tilde{\Lambda}_{2N+1}(z).$$

As a consequence we have

$$\tilde{F}(z) = \frac{\tilde{\Lambda}_{2N+1}^*(z)}{\tilde{\Psi}_{2N+1}^*(z)} = \frac{\Lambda_{2N}^*(z) - \bar{\gamma} z \Lambda_{2N}(z)}{\Psi_{2N}^*(z) + \bar{\gamma} z \Psi_{2N}(z)}$$

$$= \frac{\Omega_N^*(z^2) - \bar{\gamma} z \Omega_N(z^2)}{\Phi_N^*(z^2) + \bar{\gamma} z \Phi_N(z^2)} = \frac{\Omega_k^*(z^2) - \bar{\gamma} z^{2(N-k)+1} \Omega_k(z^2)}{\Phi_k^*(z^2) + \bar{\gamma} z^{2(N-k)+1} \Phi_k(z^2)}.$$

If $\tilde{F}$ has a pole α on $\mathbb{T}$, then

$$|\Phi_k^*(\alpha^2)| = |\bar{\gamma}\alpha^{2(N-k)+1}\Phi_k(\alpha^2)|$$

with $\alpha = e^{i\theta}$. Then, from $|\Phi_k^*(e^{2i\theta})| = |\Phi_k(e^{2i\theta})|$, we deduce $|\gamma| = 1$, which contradicts $|\gamma| < 1$. So, the corresponding orthogonality measure is absolutely continuous. (See [7].)

Notice that the Bernstein–Szegő class is preserved under such a transformation of the Schur parameters.

Furthermore, in the positive definite case, the Szegő function is

$$D(z; d\mu) = \frac{\kappa_k}{\Phi_k^*(z)}.$$

The modified Szegő function can be written as

$$\begin{aligned} D(z; d\tilde{\nu}) &= \frac{\tilde{\kappa}_{2N+1}}{\tilde{\Psi}_{2N+1}^*(z)} \\ &= \frac{(1-|\gamma|^2)^{-1/2}\kappa_N}{\Psi_{2N}^*(z) + \bar{\gamma} z \Psi_{2N}(z)} = \frac{(1-|\gamma|^2)^{-1/2}\kappa_k}{\Phi_k^*(z^2) + \bar{\gamma} z^{2(N-k)+1}\Phi_k(z^2)}. \end{aligned}$$

Notice that when $k = 0$, (Lebesgue measure) we have

$$\tilde{F}(z) = \frac{1 - \bar{\gamma} z^{2N+1}}{1 + \gamma z^{2N+1}}.$$

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